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Equivalence between angular spectrum-based and multipole expansion-based formulas of the acoustic radiation force and torque[a]

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ABSTRACT:
Two main methods have been proposed to derive the acoustical radiation force and torque applied by an arbitrary acoustic field on a particle: The first one relies on the plane wave angular spectrum decomposition of the incident field (see Sapozhnikov and Bailey [J. Acoust. Soc. Am. **133**, 661–676 (2013)] for the force and Gong and Baudoin [J. Acoust. Soc. Am. **148**, 3131–3140 (2020)] for the torque), while the second one relies on the decomposition of the incident field into a sum of spherical waves, the so-called multipole expansion (see Silva [J. Acoust. Soc. Am. **130**, 3541–3544 (2011)] and Baresch, Thomas, and Marchiano [J. Acoust. Soc. Am. **133**, 25–36 (2013)] for the force, and Silva, Lobo, and Mitri [Europhys. Lett. **97**, 54003 (2012)] and Gong, Marston, and Li [Phys. Rev. Appl. **11**, 064022 (2019)] for the torque). In this paper, we formally establish the equivalence between the expressions obtained with these two methods for both the force and torque.
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I. INTRODUCTION

Since the seminal works of Rayleigh,[1,2] Langevin,[3,4] and Brillouin,[5,6] many expressions of the acoustic radiation force and torque applied by various acoustic fields on different types of particle have been derived. King[7] was the first to propose an expression of the acoustic radiation force applied on a rigid sphere by a plane (standing or progressive) wave. This expression was extended later on by Yosika and Kawasima[8] for compressible particle and Hasegawa and Yiosika[9] for an elastic sphere. The case of spherical and focused incident waves was addressed by Embleton[10] and Chen and Apfel[11] for rigid and elastic spheres, respectively. Nevertheless, all these cases assume axisymmetric incident fields centered on the particle, which considerably simplifies the problem and does not enable one to compute the three-dimensional (3D) trapping force applied by a selective tweezer on an object.[12] The case of arbitrary acoustic field was at this point only treated in the framework of the long wavelength regime (LWR), i.e., for a particle much smaller than the wavelength.[13] Concerning the torque, the very existence of a torque applied on a spherical particle requires the existence of a momentum carried out by the wave, which cannot be obtained with an axisymmetric acoustic field. Busse and Wang[14] demonstrated the role played by the viscous boundary layer on the torque applied by orthogonal acoustic waves on a spherical particle in the LWR. Later on, Zhang and Marston[15] proposed an expression of the axial acoustic radiation torque acting on an axisymmetric particle centered on the axis of a cylindrical acoustical vortex. But again, the proposed expressions assume certain symmetry of the incident beam and specific location of the scatterer.

The treatment of the general problem of the acoustic radiation force and torque applied on a spherical particle of arbitrary size requires one to solve three major issues: First, the incident field must be decomposed into a sum of elementary waves suitable for the treatment of the scattering problem and then the calculation of the force and torque. In the angular spectrum method (ASM),[16,17] the incident field is decomposed into a sum of plane waves assuming prior knowledge of the incident field in a source plane. In the multipole expansion method (MEM) the incident field is decomposed into a sum of spherical waves,[18–22] whose respective weight (the beam-shape coefficients) can be calculated by different methods.[20,23–28]

Second, the scattering problem must be solved. For an arbitrary wave, this task is complexified by the non-axisymmetry of the incident acoustic field. The ASM alleviates the problem by using the fact that the solution of the scattering problem for a plane wave is known. Nevertheless, each plane wave of the angular spectrum decomposition has a different incident angle. This problem was solved by Sapozhnikov and Bailey[16] by using the Legendre addition theorem. In the multipole expansion method, the scattering

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problem was solved for an arbitrary spherical wave. It was shown by Baresch *et al.*[20] that the problem degenerates to the one of an incident plane wave so that the classical scattering coefficients can be used (see Appendix A in Ref. 20). Third, the force and torque can be calculated by integrating the time-averaged linear and angular radiation stress tensor over the particle surface, respectively. Such integration over the particle surface can be tedious to perform directly since (i) the particle surface is vibrating and hence is varying over time, (ii) the particle geometry may be complex in the case of non-spherical particles, and (iii) the existence of viscous and thermal boundary layers must be considered in the near field. It was first shown by Brillouin[5,6] that the integral over the vibrating surface of the particle can be transferred to a still surface by replacing the stress tensor by the so-called Brillouin tensor, which includes the momentum flux through this steady surface. Later on, it was shown that the integral can be transferred to a closed surface in the far field by using the momentum[29,30] and angular momentum balances[31–33] in the surrounding fluid and the Gauss divergence theorem[29] or the Reynolds transport theorem.[34] Hence, by choosing a spherical surface in the far field, (i) the integration is conducted over a simpler surface (concentric with the particle center) and (ii) the far-field approximation enables one to use asymptotic expressions for Bessel and Hankel functions which simplifies the integration procedure. This also enables the treatment of non-spherical particles, e.g., using T-matrix method.[22] Note that (i) one must be cautious when performing the far-field calculation of the acoustic radiation force as pointed out recently[35,36] and (ii) that the calculation can be performed without invoking the far-field approximation.[36]

Of course, the values of the acoustic radiation force and torque must be independent of the method used to calculate them. While some links between some of the expressions of the ARF available in the literature have been previously evoked,[12,37] there is no explicit demonstration of the link between these complex formulas. The present paper aims at clarifying this point and formally establishing the equivalence between the different expressions of the ARF and ART derived with different approaches.

II. DECOMPOSITION OF THE INCIDENT FIELD

In the multipole expansion method (MEM),[18–22] the incident acoustic potential is directly decomposed in the spherical waves basis as follows:

$$\Phi_i = \Phi_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m j_n(kr) Y_n^m(\theta,\varphi) e^{-i\omega t}, \tag{1}$$

with a_n^m the incident *beam-shape coefficients* (BSC), which set the weight of each spherical wave, and $Y_n^m(\theta,\varphi)$ the normalized spherical harmonics defined by

$$Y_n^m(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\varphi}, \tag{2}$$

with (r,θ,φ) the spherical coordinates, Φ_0 the potential amplitude, j_n the Bessel function of the first kind, k the wavenumber, and P_n^m the associated Legendre functions. Note that only the Bessel functions of the first kind appear in this expression since the incident field exists in the absence of the scatterer and hence must be finite in $(r=0)$, hence eliminating the Bessel functions of the second kind which are singular at this point. Different methods can be used to determine the BSC including analytical methods for ideal fields such as off-axis cylindrical Bessel beams,[26,27] quasi-analytical methods based on the translation-addition theorem,[23,25] and numerical methods based on the translation and rotation matrices[20] when the beam-shape coefficients of the incident field are known in a specific reference frame or on numerical quadrature such as the Gauss-Legendre[24] or the Lebedev quadrature[28] for an arbitrary field. Note that in this last case, the field must be known on a spherical surface surrounding the particle.

In the ASM,[16,17] the calculation starts from the prior knowledge of the incident pressure field in a source plane $(z=0)$ $p_i|_{z=0} = p_i(x,y,0)$, and its decomposition into a sum of plane waves,

$$p_i(x,y,z) = \frac{1}{4\pi^2}\iint_{k_x^2+k_y^2\le k^2} S(k_x,k_y) e^{ik_x x + ik_y y + i\sqrt{k^2-k_x^2-k_y^2}z} dk_x dk_y, \tag{3}$$

using the angular spectrum decomposition (2D spatial Fourier transform) of the source plane field,

$$S(k_x,k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_i(x,y,0) e^{-ik_x x - ik_y y} dx dy\,, \tag{4}$$

with k_x and k_y are the lateral components of the wavenumber $\mathbf{k}$ in Cartesian coordinates, $k_z^2 = k^2 - k_x^2 - k_y^2$ and $k = \omega/c$.

If the angle γ between the position vector $\mathbf{r} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}$ and the wavevector $\mathbf{k}$ is introduced, we see clearly that the incident field is nothing but the sum of plane waves $p_i^{\mathbf{k}}$ with different incident angles γ,

$$p_i^{\mathbf{k}}(x,y,z) = S(k_x,k_y) e^{ikr\cos(\gamma)}.$$

Using (i) the known decomposition of a plane wave with an incident angle γ into spherical waves and (ii) the Legendre addition theorem to express the final result as a function of the absolute spherical coordinate (θ,φ) instead of the auxiliary angle γ, Sapozhnikov and Bailey[16] were able to express the incident field into a sum of spherical waves,

$$p_i = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n H_{nm} j_n(kr) Y_n^m(\theta,\varphi), \tag{5}$$

with the ASM-based BSC H_{nm} describing the respective weight of each spherical wave,

$$H_{nm} = \iint_{k_x^2+k_y^2\le k^2} S(k_x,k_y)\left[Y_n^m(\theta_k,\varphi_k)\right]^* dk_x dk_y. \tag{6}$$

The asterisk designates the complex conjugate and the angle parameters (θ_k, φ_k) in the Fourier space have the relation: $\cos\theta_k = [1-(k_x^2+k_y^2)/k^2]^{1/2}$ and $\varphi_k = \arctan(k_y/k_x)$. This decomposition into a sum of spherical waves is necessary to compute the force and torque since the total field (incident + scattered) needs to be integrated over an arbitrary closed surface surrounding the particle, which for commodity will be chosen as a spherical surface in the far field as discussed in Sec. IV. The H_{nm} coefficients can be easily obtained when the field is known in a source plane by using the spatial fast Fourier transform of the incident field,[38,39] which makes the ASM method very convenient to compute the force applied on a particle by a field generated by a planar transducer.[39–43]

The comparison of Eqs. (1) and (5) and use of the relationship between the velocity potential and pressure $p_i = i\omega\rho_0\Phi_i$ (with ω the angular frequency and ρ_0 the fluid density), enables one to establish the relationship between the incident BSC a_n^m and the angular spectrum-based BSC H_{nm},

$$a_n^m = \frac{1}{\pi\omega\rho_0\Phi_0} i^{n-1} H_{nm}, \tag{7}$$

which is essential to prove the equivalence of MEM and ASM based ARF and ART formulas. Note that an equivalent form of Eq. (7) has been given in Eq. (15) of Ref. 28 by comparing two expressions of acoustic pressure.

III. RESOLUTION OF THE SCATTERING PROBLEM

In the MEM, the scattered field, as the incident field is decomposed directly into a sum of spherical waves,

$$\Phi_s = \Phi_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} s_n^m h_n^{(1)}(kr) Y_n^m(\theta,\varphi) e^{-i\omega t}, \tag{8}$$

with s_n^m the beam-shape coefficient of the scattered field. Note that this time the scattered field is expressed in terms of the Hankel function of the first kind since the scattered field is an outgoing wave, hence eliminating the Hankel function of the second kind (corresponding to converging wave in the convention used here for the temporal part of the wave $e^{-i\omega t}$). The expression of the scattered beam-shape coefficients as a function of the incident beam-shape coefficients requires to solve the scattering problem, i.e., to determine the *partial wave coefficients* A_n^m defined by $s_n^m = A_n^m a_n^m$. These coefficients depend on the particle shape, material composition, and surface boundary condition. In the MEM, the solution of the scattering problem is *a priori* not known since the axisymmetry and resulting simplifications used in the case of plane waves can no longer be invoked. The complete problem was solved by Baresch *et al.*[20] for an elastic sphere through the introduction of three scalar potentials (one for the longitudinal wave and the two Debye potentials for the shear wave, solutions of the wave equation, and then applying the boundary conditions). It was shown that, in fact, the problem degenerates to the one of plane incident wave, so that the partial waves coefficient A_n computed for a plane wave, which do not depend on the index m, can be used. Note that in this simplified case, people sometime introduce the so-called *scattering coefficients* S_n linked to the partial wave coefficients by the formula $A_n = (S_n - 1)/2$. Also note that in the general case of non-spherical particles, the partial wave coefficients can be determined using the transition matrix method[21,22,44] which makes the theory operable for non-spherical shapes, such as spheroids[45] and finite cylinders.[46,47]

In the ASM, the treatment relies on known results for the scattering of a plane wave by a sphere. Indeed, (i) the incident field has been decomposed into a sum of plane waves and (ii) the solution of the scattering problem is known for each plane wave. Hence, using these solutions for each plane wave and then using (i) the decomposition of a plane wave into a sum of spherical waves and (ii) the Legendre addition theorem, the scattered field can also be decomposed into a sum of spherical waves,

$$p_s = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n H_{nm} A_n^m h_n^{(1)}(kr) Y_n^m(\theta,\varphi). \tag{9}$$

IV. CALCULATION OF THE FORCE AND TORQUE

The last step, which is common to ASM and MEM is to compute the integral of the stress tensor or angular stress tensor over the surface of the particle to compute the force and torque, respectively. One major difficulty comes from the fact that the surface of the particle is vibrating. This problem can be overcome in two ways: first, using Lagrangian coordinates instead of Eulerian coordinates and, second, transferring the integral to a still surface by subtracting the flux of momentum (flux of angular momentum) to the stress tensor (angular stress tensor) for the force and torque, respectively, as first demonstrated by Brillouin (for the force).[5,6] To simplify the calculation, these integrals can be transported to any surface surrounding the particle, e.g., for simplicity a spherical surface in the far field as demonstrated by Westervelt for the force[29,30] and Maidanik and others for the torque.[31–33] Using these results, the integrals to compute the force $\mathbf{F}$ and torque $\mathbf{T}$ can be written under the following form in terms of the acoustic potential $(\Phi_{i,s})$ of the incident and scattered field as

$$\mathbf{F} = \frac{\rho_0 k^2}{2} \iint_{S_0} \mathrm{Re}\left\{ \left(\frac{i}{k}\frac{\partial \Phi_i}{\partial r} - \Phi_i \right) \Phi_s^* - \Phi_s \Phi_s^* \right\} \mathbf{n} dS, \tag{10}$$

$$\mathbf{T} = \frac{\rho_0}{2} \mathrm{Im}\left\{ \iint_{S_0} \left(\frac{\partial \Phi_i^*}{\partial r} \mathbf{L}\Phi_s + \frac{\partial \Phi_s^*}{\partial r} \mathbf{L}\Phi_i + \frac{\partial \Phi_s^*}{\partial r} \mathbf{L}\Phi_s \right) dS \right\}, \tag{11}$$

where S_0 is a closed spherical surface in the far field centered at the mass center of the particle, ρ_0 is the density at rest, "Re" means the real part of a complex number, "Im" designates the imaginary part, $\mathbf{n}$ is the outward unit normal

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vector, and the differential surface area is $dS = r^2 \sin\theta d\theta d\varphi$ with θ and φ the polar and azimuthal angles, $\mathbf{L} = -i(\mathbf{r} \times \nabla)$ is the angular momentum operator, with its components in the three directions $L_{x,y,z}$, and the recursion relations of the normalized spherical harmonics with ladder operators $L_{\pm}$ given in detail in Appendix D.

In Sec. V we establish the link between the different formulas obtained in the literature.

V. EQUIVALENCE OF THE THREE-DIMENSION ARF FORMULAS

Expressions of the ARF exerted by an arbitrary field on an arbitrary located spherical scatterer have been established independently by three different groups: Silva[18] and Baresch *et al.*[20] with a MEM and Sapozhnikov and Bailey based on the ASM.[16] The equivalence between the formulas obtained by Baresch *et al.*[20] and Sapozhnikov and Bailey[16] has been briefly discussed by Thomas *et al.*[12,37] while the equivalence with Silva's work has not been investigated yet. In this section, the reason for the different forms of ARF formulas by Silva[18] and Baresch *et al.*[20] is provided (since both use the MEM), while pointing out some minor existing issues in the formula and at the same time, for the first time, providing detailed proof of the equivalence of the ARF formulas for the three works.

A. Equivalence between MEM formula and compact formulation

1. MEM formula by Silva and Gong et al. and reindexing

Following the work of Silva[18] and of Gong *et al.*,[21] the dimensionless ARF formulas in terms of the incident a_n^m and scattered s_n^m BSC are obtained by substituting Eq. (1) and (8) into Eq. (10) and conducting several algebraic calculations given in Eqs. (11)–(13) of Ref. 18 by Silva or Eqs. (12)–(14) of Ref. 21 by Gong *et al.* The ARF formulas can be therefore obtained based on the relation with the dimensionless ARF according to Eq. (10) in Ref. 21. Note that for the two separate derivations, different asymptotic expressions of velocity potentials in the far-field are used: Silva uses trigonometric functions[18] [see Eq. (4) in his paper], while Gong *et al.* use the exponential functions[21] to approximate the Bessel and Hankel functions. In addition, the work of Gong *et al.* work is an extension of numerical implementation for non-spherical shapes by using the T-matrix method.[21]

However, the ARF formulas by Silva[18] and Gong *et al.*[21] missed a re-indexing step in the scattered BSC (s_{n-1}^{m+1}, s_{n-1}^{m-1}, and s_n^{-m-1}), as pointed out recently.[12,37] Here, we explain the reason for the index issue and provide the good expressions: Silva and Gong *et al.* use the simplified double summation symbol $\sum_{nm}$ to represent $\sum_{n=0}^{\infty}\sum_{m=-n}^{n}$ to conduct the integral process involving the product of two spherical harmonics [see Eq. (11) in Ref. 21]. A mistake appears since the regime of m should be correctly chosen for the spherical harmonics $Y_{n-1}^{m\pm1}$ and Y_{n-1}^{m} [as given in Eqs. (A2) and (A4)] based on the definition of spherical harmonics Y_n^m with $|m| \le n$, which means $\sum_{nm}$ is not always $\sum_{n=0}^{\infty}\sum_{m=-n}^{n}$.

In this work, we re-derive the formulas following the right indexes (n, m) and therefore get the correct forms as (see details in Appendix B)

$$F_x = \frac{\rho_0\Phi_0^2}{4}\mathrm{Im}\Bigg\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[b_{n+1}^{-m}\Big[\big(a_n^m + s_n^m\big)s_{n+1}^{m-1*} - \Big(a_{n+1}^{m-1} + s_{n+1}^{m-1}\Big)s_n^{m*}\Big] + b_{n+1}^{m}\Big[\Big(a_{n+1}^{m+1} + s_{n+1}^{m+1}\Big)s_{nm}^{*} - \big(a_n^m + s_n^m\big)s_{n+1}^{m+1*}\Big]\Big]\Bigg\}, \quad (12a)$$

$$F_y = \frac{\rho_0\Phi_0^2}{4}\mathrm{Re}\Bigg\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[b_{n+1}^{-m}\Big[\big(a_n^m + s_n^m\big)s_{n+1}^{m-1*} + \Big(a_{n+1}^{m-1} + s_{n+1}^{m-1}\Big)s_n^{m*}\Big] + b_{n+1}^{m}\Big[\Big(a_{n+1}^{m+1} + s_{n+1}^{m+1}\Big)s_n^{m*} + \big(a_n^m + s_n^m\big)s_{n+1}^{m+1*}\Big]\Big]\Bigg\}, \quad (12b)$$

$$F_z = \frac{\rho_0\Phi_0^2}{2}\mathrm{Im}\Bigg\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}c_{n+1}^{m}\Big[\big(a_{n+1}^{m*} + s_{n+1}^{m*}\big)s_n^m + \big(a_n^m + s_n^m\big)s_{n+1}^{m*}\Big]\Bigg\}, \quad (12c)$$

where $n \in [0, \infty]$ and $m \in [-n, n]$, and the coefficients b_n^m and c_n^m defined in terms of n and m are given in Appendix A.

Note that despite the index issue, the numerical computations in Ref. 21 are correct since the erroneous additional terms were cancelled in the numerical procedure. This can be further verified by the comparison of results by Gong *et al.* with the partial wave based results by Marston.[48] Note also that this set of formulas can be written in a much more compact form using the relation $s_n^m = A_n^m a_n^m$, which will be given in Sec. V A 2.

2. The ARF formulas by Baresch et al.

Another set of ARF formulas based on the MEM, was derived by Baresch *et al.* for 3D ARF on an arbitrarily located elastic sphere, as given by Eqs. (14)–(16) in Ref. 20 [reorganized as Eqs. (1)–(3) by Zhao *et al.* in Ref. 28]. Note that there is a typo for the regime of index m (Refs. 12, 20, and 28): it should be $|m| \le n$ instead of $|m| < n$ (otherwise the formulas are not equivalent to those by Sapozhnikov and Bailey[16]) The ARF formulas with the right index regimes by Thomas and colleagues are equivalent to the above-corrected version [see Eq. (12) of Silva[18] and the formulas of Gong *et al.*[21]].

The difference between formulas by Silva[18] (or Gong *et al.*[21]) and Baresch *et al.*[20] are the following. (i) Silva uses the incident a_n^m and scattered s_n^m BSC. Baresch *et al.* solved the scattering problem for an elastic sphere insonified by an arbitrary incident beam and showed that the problem degenerates to the one of the scattering of an incident plane wave, so that the corresponding partial wave coefficients A_n can be used leading to the relation $s_n^m = A_n a_n^m$. (ii) Silva uses the orthogonality and recursion relationship of normalized spherical harmonics directly based on Arfken's textbook (see Appendix A),[49] while Baresch *et al.* use the orthogonality relationship of associated Legendre functions (P_n^m) and exponential functions, and also the recursion relationship of associated Legendre functions. This leads to the fact that there are four terms for the lateral forces and two terms for the axial force in Silva's work (without reindexing),[18] while only two terms for the lateral and one term for the axial forces by Zhao *et al.*[28] (with reindexing during the derivation procedure).[20] (iii) Silva uses the normalized spherical harmonics, while Baresch *et al.* use the unnormalized spherical harmonics to derive the ARF formulas,[20] which have been re-organized with normalized spherical harmonics to be compact by Zhao *et al.* [Eqs. (1)–(3)].[28]

3. Compact expression of the ARF for arbitrary shaped particles

If we substitute the relation $s_n^m = A_n^m a_n^m$ for a particle with an arbitrary shape, the correct version of ARF formulas in terms of a_n^m and s_n^m in Eq. (12) is further written in a compact manner as

$$F_x = \frac{\rho_0 \Phi_0^2}{4} \mathrm{Im}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(C_n^{m-1} b_{n+1}^{-m} a_n^m a_{n+1}^{m-1*} - C_n^{m+1} b_{n+1}^{m} a_n^m a_{n+1}^{m+1*} \right) \right\}, \quad (13a)$$

$$F_y = \frac{\rho_0 \Phi_0^2}{4} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(C_n^{m-1} b_{n+1}^{-m} a_n^m a_{n+1}^{m-1*} + C_n^{m+1} b_{n+1}^{m} a_n^m a_{n+1}^{m+1*} \right) \right\}, \quad (13b)$$

$$F_z = \frac{\rho_0 \Phi_0^2}{2} \mathrm{Im}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m c_{n+1}^m a_n^m a_{n+1}^{m*} \right\}, \quad (13c)$$

with $C_n^{m\mp1} = A_n^m + 2A_n^m A_{n+1}^{m\mp1*} + A_{n+1}^{m\mp1*}$ and $C_n^m = A_n^m + 2A_n^m A_{n+1}^{m*} + A_{n+1}^{m*}$. These compact equations (13) are equivalent to the re-organized ones (using normalized spherical harmonics instead of spherical harmonics in Ref. 20) by Zhao *et al.* in a direct way[28] when the particle shape is considered as a sphere (so that $A_n^m = A_n$ and $C_n^{m\mp1} = C_n^m = C_n = A_n + 2A_nA_{n+1}^* + A_{n+1}^*$) and the index m is with the right regime $|m| \le n$.

B. Equivalence analysis of the three sets of ARF formulas

As claimed above, the different forms of ARF formulas derived by Thomas and colleagues[20,28] (compact form of correct version of ARF formulas by Silva[18] and Gong *et al.*[21]) and Sapozhnikov and Bailey[16] come from the different elementary wave expansion of velocity potential or pressure. The explicit relation between the beam coefficient a_n^m based on MEM and H_{nm} based on ASM is given by Eq. (7) in Sec. II, which can be used to substitute into Eq. (13) to derive the 3D ARF formulas in terms of the notation H_{nm} introduced by Sapozhnikov and Bailey based on the ASM. The equivalence between the two sets of formulas will be verified immediately if $A_n^m = A_n$ is set for a spherical shape (see details in Appendix C). The question raised by Sapozhnikov and Bailey in their paper[16] between their formula and the one by Silva is now solved. All in all, considering the correction of the index issues pointed out above, all the three sets of original 3D ARF formulas are proved to be equivalent.

VI. EQUIVALENCE OF THREE-DIMENSIONAL ART FORMULAS

The ART on a particle in an ideal fluid can be calculated by the integral of the time-averaged angular stress tensor minus the angular momentum flux over a far-field standard spherical shape centered at the mass center of the particle[15,19,22,31] [see Eq. (11)]. Explicit expressions of 3D ART formulas have been derived by Silva *et al.*[19] and Gong *et al.*[22] based on the MEM and Gong and Baudoin[17] based on the ASM.

As for the ART, there are also index issues in the expression obtained by Silva *et al.*[19] and Gong *et al.*[22] Here, we provide the correct expressions of ART formulas by Silva *et al.*[19] and Gong *et al.*[22] based on the multipole expansion method (see details in Appendix E):

$$T_x = -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} \bar{b}_n^m \left[\left(a_n^{m*} + s_n^{m*}\right) s_n^{m-1} + \left(a_n^{m-1*} + s_n^{m-1*}\right) s_n^m \right] \right\}, \quad (14a)$$

$$T_y = -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Im}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} \bar{b}_n^m \left[\left(a_n^{m*} + s_n^{m*}\right) s_n^{m-1} - \left(a_n^{m-1*} + s_n^{m-1*}\right) s_n^m \right] \right\}, \quad (14b)$$

$$T_z = -\frac{\rho_0 \Phi_0^2}{2k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} m \left(a_n^{m*} + s_n^{m*}\right) s_n^m \right\}, \quad (14c)$$

with the coefficients $\bar{b}_n^m$ given in Appendix D. Again, it is noteworthy that the numerical computations in Ref. 22 are correct since they use the definition for the scattered BSC that $s_n^m = 0$ when $n < 0$ or $|m| > n$.

The relationship $s_n^m = A_n^m a_n^m$ can be introduced into Eq. (14) to obtain a set of compact formulas in terms of the incident BSC only,

$$T_x = -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} \bar{b}_n^m \bar{C}_n^m a_n^{m*} a_n^{m-1} \right\}, \quad (15a)$$

$$T_y = -\frac{\rho_0\Phi_0^2}{4k}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n} \bar{b}_n^m \bar{C}_n^m a_n^{m*} a_n^{m-1}\right\}, \qquad (15b)$$

$$T_z = -\frac{\rho_0\Phi_0^2}{2k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} m\bar{D}_n^m a_n^{m*} a_n^m\right\}, \qquad (15c)$$

where $\bar{C}_n^m = A_n^{m-1} + A_n^{m*} + 2A_n^{m-1}A_n^{m*}$, $\bar{D}_n^m = A_n^m + A_n^m A_n^{m*}$. The above compact ART formulas are identical to Eqs. (10)–(12) of Ref. 17 by using the relation between a_n^m and H_{nm} given by Eq. (7) in Sec. II (see details in Appendix F). Hence, the equivalence of the ART formulas between the correct form [see Eq. (14)] of the work of Silva *et al.* and Gong *et al.* based on the MEM[19,22] and those derived by Gong and Baudoin based on the ASM[17] has been demonstrated in this section.

VII. CONCLUSIONS AND DISCUSSIONS

In summary, we provide in this paper a clear proof of the equivalence of the three sets of the 3D acoustic radiation force (ARF) formulas derived independently by Silva[18] (extended later on by Gong *et al.*[21] to arbitrary shape particles), Thomas and associates,[20,28] and Sapozhnikov and Bailey,[16] and the 3D acoustic radiation torque (ART) formulas derived by Silva *et al.*[19] (extended by Gong *et al.*[22] to arbitrary shape particles) and Gong and Baudoin.[17] The reasons for the different forms of ARF and ART expressions are discussed completely in Secs. V and VI, respectively.

The advantage of the MEM-based ARF and ART formulas[18,20,21] is that the calculations of 3D ARF and ART are direct by using the incident BSC a_n^m of known acoustic fields which has a long research history in the literature for scattering problems. For arbitrary fields, numerical methods based on numerical quadrature[24,28] require prior knowledge of the incident field on a spherical surface surrounding the particle, which can be difficult to set in practice experimentally. The advantage of the ASM-based ARF[16] and ART[17] is that they are easy to set up when the field is known (i.e., measured) in a transverse plane, which is convenient for planar holographic transducers.[38–43] Note that this set of formulas can also be used for ideal beams whose introduced coefficients H_{nm} are available either by using the angular spectrum of the beam $S(k_x, k_y)$[16] or the relation given in Eq. (7).

To finalize the calculation of the ARF and ART with all these formulas, the key point is to obtain the partial wave coefficients A_n^m of the particle exactly. Silva,[18,19] Thomas and colleagues,[20,28] and Sapozhnikov and Bailey[16] discuss particles with spherical shapes so that A_n^m only depends on the index n, having $A_n^m = A_n$. Gong *et al.* derive the formulas with A_n^m depending on the indexes of (n, m) with several numerical computation for arbitrary-sized non-spherical shapes by a semi-analytical T-matrix method.[21,22,44] For a rigid spheroidal particle in the so-called long-wavelength limit, Silva and colleagues gives the A_n^m with the Taylor expansion up to the dipole $(n=1)$ in spheroidal coordinates[50] and obtain concise analytical ARF and ART expressions using the partial wave expansion.[51] Note also that the overall formulas discussed in the present work are generally applied for a particle in an ideal fluid but are still applicable for a particle in a viscous fluid if the viscous effect in the fluid can be accounted in the expression of scattering (partial wave) coefficients.[15,52]

From a perspective viewpoint, the present work on the ARF and ART formulas may be extended for multiple particles[53–55] if the partial wave coefficients are available, which can be used for the manipulation and assembly of large particles beyond Rayleigh regime.[56,57] Based on Eqs. (13) and (15), the ARF and ART are closely related to the scattering from the particle in a fluid. Hence, the scattering characteristics are essential to the acting force and torque of acoustic field on the particle. For example, the resonance scattering from an elastic sphere may be suppressed under an on-axis Bessel beam of selected parameters and be not with an off-axis incidence,[58–60] which could be used to tune the ARF and ART, such as a stable tractor (pulling) beam,[48,61,62] or a 3D stable trapping[23,63] with suppressed spinning rotation. More importantly, the present work will help to build an acoustical tweezers numerical toolbox[44,64] as an analogy to its optical counterpart.[65]

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APPENDIX A: ORTHOGONALITY AND RECURRENCE RELATIONS OF SPHERICAL HARMONICS

The orthogonality relationship of normalized spherical harmonics is given in Eq. (15.138) by Arfken *et al.*,[49]

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_n^{m*} Y_{n'}^{m'} = \delta_{nn'}\delta_{mm'}. \qquad (A1)$$

The recurrence relations of normalized spherical harmonics involved with trigonometric and exponential functions are given in Eqs. (15.150) and (15.151) by Arfken *et al.*,[49] respectively,

$$\cos\theta Y_n^m = c_n^m Y_{n-1}^m + c_{n+1}^m Y_{n+1}^m, \qquad (A2)$$

with

$$c_n^m = \sqrt{\frac{(n+m)(n-m)}{(2n-1)(2n+1)}}, \qquad (A3)$$

which is based on a recurrence relation of associated Lengendre functions [Eq. (15.88) in the textbook of Arfken *et al.*], as also used by Baresch *et al.*[20] in Eq. (C5) in Appendix C and

$$e^{\pm i\varphi}\sin\theta Y_n^m = \pm b_n^{\mp m-1} Y_{n-1}^{m\pm 1} \mp b_{n+1}^{\pm m} Y_{n+1}^{m\pm 1} \qquad (A4)$$

with

$$b_n^m = \sqrt{\frac{(n+m)(n+m+1)}{(2n-1)(2n+1)}}, \tag{A5}$$

which is based on two recurrence relations of associated Lengendre functions [Eqs. (15.89) and (15.90) in the textbook of Arfken *et al.*], with Eq. (15.89) also used by Baresch *et al.*[20] in Appendix D.

By using the Euler's formula $e^{\pm i\varphi} = \cos\varphi \pm i \sin\varphi$, the terms of normalized spherical harmonics involved with trigonometric functions ($\cos\varphi \sin\theta Y_{nm}$ and $\sin\varphi \sin\theta Y_{nm}$) can be obtained, which can be further applied into Eq. (11) in Gong *et al.*[21] for the final 3D ARF expressions. The relation used for the derivation of F_x is

$$\begin{aligned}2\cos\varphi \sin\theta Y_n^m = b_n^{-m-1} Y_{n-1}^{m+1} - b_{n+1}^m Y_{n+1}^{m+1}\\ + b_{n+1}^{-m} Y_{n+1}^{m-1} - b_n^{m-1} Y_{n-1}^{m-1}\end{aligned} \tag{A6}$$

and the expression for the derivation of F_y is

$$\begin{aligned}2i \times \sin\varphi \sin\theta Y_n^m = b_n^{-m-1} Y_{n-1}^{m+1} - b_{n+1}^m Y_{n+1}^{m+1}\\ - b_{n+1}^{-m} Y_{n+1}^{m-1} + b_n^{m-1} Y_{n-1}^{m-1}.\end{aligned} \tag{A7}$$

APPENDIX B: DETAILED DERIVATION OF ARF WITH CORRECT INDEX

1. Detailed derivation of F_x

Based on the ARF formulas of Eq. (9) from Ref. 21, the expression of the x-component of ARF is

$$\begin{aligned}F_x &= \frac{1}{2}\rho_0 k^2 \Phi_0^2 \iint_{S_0} \mathrm{Re}\left\{-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} \frac{i^{n'-n}}{(kr)^2}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} Y_n^m Y_{n'}^{m'*}\right\} r^2 \sin\theta \cos\varphi \sin\theta d\theta d\varphi\\ &= -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} Y_n^m Y_{n'}^{m'*} \sin\theta \cos\varphi \sin\theta d\theta d\varphi\right\}.\end{aligned} \tag{B1}$$

Substituting Eq. (A6) into Eq. (B1), F_x can be divided into four terms,

$$\begin{aligned}F_x = -\frac{1}{4}\rho_0 \Phi_0^2 \mathrm{Re}\Bigg\{&\sum_{n=1}^{\infty}\sum_{m=-n}^{n-2}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} b_n^{-m-1} Y_{n-1}^{m+1} Y_{n'}^{m'*} \sin\theta d\theta d\varphi\\ &+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} - b_{n+1}^m Y_{n+1}^{m+1} Y_{n'}^{m'*} \sin\theta d\theta d\varphi\\ &+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} b_{n+1}^{-m} Y_{n+1}^{m-1} Y_{n'}^{m'*} \sin\theta d\theta d\varphi\\ &+\sum_{n=1}^{\infty}\sum_{m=-n+2}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} - b_n^{m-1} Y_{n-1}^{m-1} Y_{n'}^{m'*} \sin\theta d\theta d\varphi\Bigg\}.\end{aligned} \tag{B2}$$

It is important to note that the index regimes of (n, m) for different terms are different because the correct index regime of Y_n^m should be $n \in [0,\infty]$ and $m \in [-n, n]$ for the indexes. In addition, based on the definition in Eq. (1), the intersection of regime of (n, m) is listed in Table I.

Using the orthogonality relation of the normalized spherical harmonics in Eq. (A1), Eq. (B2) can be further written as

TABLE I. Regime of (n, m) in normalized spherical harmonics for derivation of F_x and F_y. Note that based on the definition in Eq. (1), we have $n \in [0,\infty]$ and $m \in [-n, n]$.

	n	m	Intersection
Y_{n-1}^{m+1}	$n \in [1,\infty]$	$m \in [-n, n-2]$	$n \in [1,\infty]$, $m \in [-n, n-2]$
Y_{n+1}^{m+1}	$n \in [-1,\infty]$	$m \in [-n-2, n]$	$n \in [0,\infty]$, $m \in [-n, n]$
Y_{n+1}^{m-1}	$n \in [-1,\infty]$	$m \in [-n, n+2]$	$n \in [0,\infty]$, $m \in [-n, n]$
Y_{n-1}^{m-1}	$n \in [1,\infty]$	$m \in [-n+2, n]$	$n \in [1,\infty]$, $m \in [-n+2, n]$

$$F_x = -\frac{1}{4}\rho_0\Phi_0^2\mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=-n}^{n-2}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}b_n^{-m-1}\delta_{n-1,n'}\delta_{m+1,m'}\right.$$
$$+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} -i^{n'-n}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}b_{n+1}^{m}\delta_{n+1,n'}\delta_{m+1,m'}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}b_{n+1}^{-m}\delta_{n+1,n'}\delta_{m-1,m'}$$
$$\left.+\sum_{n=1}^{\infty}\sum_{m=-n+2}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} -i^{n'-n}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}b_n^{m-1}\delta_{n-1,n'}\delta_{m-1,m'}\right\}-\frac{1}{4}\rho_0\Phi_0^2\mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=-n}^{n-2}-i\left(a_n^m+s_n^m\right)s_{n-1}^{m+1*}b_n^{-m-1}\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}-i\left(a_n^m+s_n^m\right)s_{n+1}^{m+1*}b_{n+1}^{m}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}i\left(a_n^m+s_n^m\right)s_{n+1}^{m-1*}b_{n+1}^{-m}+\sum_{n=1}^{\infty}\sum_{m=-n+2}^{n}i\left(a_n^m+s_n^m\right)s_{n-1}^{m-1*}b_n^{m-1}\right\}$$
$$=-\frac{1}{4}\rho_0\Phi_0^2\mathrm{Im}\left\{\sum_{n=1}^{\infty}\sum_{m=-n}^{n-2}\left(a_n^m+s_n^m\right)s_{n-1}^{m+1*}b_n^{-m-1}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(a_n^m+s_n^m\right)s_{n+1}^{m+1*}b_{n+1}^{m}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}-\left(a_n^m+s_n^m\right)s_{n+1}^{m-1*}b_{n+1}^{-m}\right.$$
$$\left.+\sum_{n=1}^{\infty}\sum_{m=-n+2}^{n}-\left(a_n^m+s_n^m\right)s_{n-1}^{m-1*}b_n^{m-1}\right\}. \quad \text{(B3)}$$

Note that $\mathrm{Re}\{X\}=\mathrm{Im}\{iX\}$ with X an arbitrary complex number. Here, a re-index is applied with $p=n-1\in[0,\infty]$ for the first and fourth term of Eq. (B3)

$$F_x=-\frac{1}{4}\rho_0\Phi_0^2\mathrm{Im}\left\{\sum_{p=0}^{\infty}\sum_{m=-p-1}^{p-1}\left(a_{p+1}^m+s_{p+1}^m\right)s_p^{m+1*}b_{p+1}^{-m-1}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(a_n^m+s_n^m\right)s_{n+1}^{m+1*}b_{n+1}^{m}\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}-\left(a_n^m+s_n^m\right)s_{n+1}^{m-1*}b_{n+1}^{-m}+\sum_{p=0}^{\infty}\sum_{m=-p+1}^{p+1}-\left(a_{p+1}^m+s_{p+1}^m\right)s_p^{m-1*}b_{p+1}^{m-1}\right\}. \quad \text{(B4)}$$

Now, using a re-index for m: for the first term $q=m+1\in[-p,p]$ and for the fourth term $q=m-1\in[-p,p]$, we have

$$F_x=-\frac{1}{4}\rho_0\Phi_0^2\mathrm{Im}\left\{\sum_{p=0}^{\infty}\sum_{q=-p}^{p}\left(a_{p+1}^{q-1}+s_{p+1}^{q-1}\right)s_p^{q*}b_{p+1}^{-q}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(a_n^m+s_n^m\right)s_{n+1}^{m+1*}b_{n+1}^{m}\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}-\left(a_n^m+s_n^m\right)s_{n+1}^{m-1*}b_{n+1}^{-m}+\sum_{p=0}^{\infty}\sum_{q=-p}^{p}-\left(a_{p+1}^{q+1}+s_{p+1}^{q+1}\right)s_p^{q*}b_{p+1}^{q}\right\}$$
$$=\frac{1}{4}\rho_0\Phi_0^2\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[b_{n+1}^{-m}\left[\left(a_n^m+s_n^m\right)s_{n+1}^{m-1*}-\left(a_{n+1}^{m-1}+s_{n+1}^{m-1}\right)s_n^{m*}\right]+b_{n+1}^{m}\left[\left(a_{n+1}^{m+1}+s_{n+1}^{m+1}\right)s_n^{m*}-\left(a_n^m+s_n^m\right)s_{n+1}^{m+1*}\right]\right]\right\}, \quad \text{(B5)}$$

which is Eq. (12a) in Sec. V A 1.

2. Derivation of F_y

The expression of the y-component of ARF is

$$F_y=\frac{1}{2}\rho_0k^2\Phi_0^2\iint_{S_0}\mathrm{Re}\left\{-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\frac{i^{n'-n}}{(kr)^2}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}Y_n^m(\theta,\varphi)Y_{n'}^{m'*}(\theta,\varphi)\right\}r^2\sin\theta\sin\varphi\sin\theta d\theta d\varphi$$
$$=-\frac{1}{2}\rho_0\Phi_0^2\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}i^{n'-n}\left(a_n^m+s_n^m\right)s_{n'}^{m'*}\iint_{S_0}Y_n^mY_{n'}^{m'*}\sin\theta\sin\varphi\sin\theta d\theta d\varphi\right\}. \quad \text{(B6)}$$

The detailed derivation of F_y is similar to that for the x-component F_x by substituting Eq. (A7) and replacing Eq. (A6) into Eq. (B6), and using of the orthogonality relationship of normalized spherical harmonics of Eq. (A1). The final expression of F_y in terms of a_n^m and s_n^m is given in Eq. (12b) in Sec. V A 1, which is not given here for brevity.

TABLE II. Regime of (n, m) in normalized spherical harmonics for derivation of F_z. Note that based on the definition in Eq. (1), we have $n \in [0, \infty]$ and $m \in [-n, n]$.

	n	m	Intersection
Y_{n-1}^m	$n \in [1, \infty]$	$m \in [-n+1, n-1]$	$n \in [1, \infty]$, $m \in [-n+1, n-1]$
Y_{n+1}^m	$n \in [-1, \infty]$	$m \in [-n-1, n+1]$	$n \in [0, \infty]$, $m \in [-n, n]$

3. Detailed derivation of F_z

The expression of the z-component of ARF is

$$F_z = \frac{1}{2}\rho_0 k^2 \Phi_0^2 \iint_{S_0} \mathrm{Re}\left\{-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} \frac{i^{n'-n}}{(kr)^2}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} Y_n^m(\theta,\varphi) Y_{n'}^{m'*}(\theta,\varphi)\right\} r^2 \cos\theta \sin\theta d\theta d\varphi$$
$$= -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} Y_n^m Y_{n'}^{m'*} \cos\theta \sin\theta d\theta d\varphi\right\}. \quad \text{(B7)}$$

Substituting Eq. (A2), we have

$$F_z = -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=-n+1}^{n-1}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} c_n^m Y_{n-1}^m Y_{n'}^{m'*} \sin\theta d\theta d\varphi\right.$$
$$\left. + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} \iint_{S_0} c_{n+1}^m Y_{n+1}^m Y_{n'}^{m'*} \sin\theta d\theta d\varphi\right\}. \quad \text{(B8)}$$

For the definition of (n, m) in Eq. (1), it has $n \in [0, \infty]$ and $m \in [-n, n]$. Since Y_{n-1}^m ($n \in [1, \infty]$ and $m \in [-n+1, n-1]$) and Y_{n+1}^m ($n \in [-1, \infty]$ and $m \in [-n-1, n+1]$) are introduced here, the final regimes of indexes (n, m) are the intersection and given differently for the first and second part (see Table II).

By using Eq. (A1), the expression of F_z is

$$F_z = -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=-n+1}^{n-1}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} c_n^m \delta_{n-1,n'}\delta_{m,m'}\right.$$
$$\left. + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n'-n}\left(a_n^m + s_n^m\right) s_{n'}^{m'*} c_{n+1}^m \delta_{n+1,n'}\delta_{m,m'}\right\}$$
$$= -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=-n+1}^{n-1} i^{-1}\left(a_n^m + s_n^m\right) s_{n-1}^{m*} c_n^m + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i\left(a_n^m + s_n^m\right) s_{n+1}^{m*} c_{n+1}^m\right\}. \quad \text{(B9)}$$

Re-index for the first part of Eq. (B9) using $p = n - 1 \in [0, \infty]$ so that $m \in [-p, p]$. The final form of F_z in terms of a_n^m and s_n^m is

$$F_z = -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{p=0}^{\infty}\sum_{m=-p}^{p} i^{-1}\left(a_{p+1}^m + s_{p+1}^m\right) s_p^{m*} c_{p+1}^m + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i\left(a_n^m + s_n^m\right) s_{n+1}^{m*} c_{n+1}^m\right\}$$
$$= -\frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i\left(a_{n+1}^{m*} + s_{n+1}^{m*}\right) s_n^m c_{n+1}^m + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i\left(a_n^m + s_n^m\right) s_{n+1}^{m*} c_{n+1}^m\right\}$$
$$= \frac{1}{2}\rho_0 \Phi_0^2 \mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_{n+1}^m \left[\left(a_{n+1}^{m*} + s_{n+1}^{m*}\right) s_n^m + \left(a_n^m + s_n^m\right) s_{n+1}^{m*}\right]\right\}, \quad \text{(B10)}$$

which is Eq. (12c) in Sec. V A 1. Note that $\mathrm{Re}\{X\} = \mathrm{Re}\ \{X^*\}$ and $\mathrm{Re}\{iX\} = -\mathrm{Im}\{X\}$.

APPENDIX C: EQUIVALENCE OF EQ. (12) AND FORMULAS BY SAPOZHNIKOV AND BAILEY

By substituting Eq. (7) into Eq. (13), we can prove that the three components of ARF formulas for a sphere (with $A_n^m = A_n$) are equivalent to those in terms of H_{nm} by Sapozhnikov and Bailey [see Eqs. (46)–(48) in Ref. 16], respectively. The detailed derivations are given below. Recall that for a sphere, one has $C_n^{m\mp 1} = C_n$. The x-component of ARF,

$$F_x = \frac{\rho_0\Phi_0^2}{4}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left(b_{n+1}^{-m}a_n^m a_{n+1}^{m-1*} - b_{n+1}^{m}a_n^m a_{n+1}^{m+1*}\right)\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left[b_{n+1}^{-m}\left(i^{n-1}H_{nm}\right)\left(i^nH_{n+1,m-1}\right)^* - b_{n+1}^{m}\left(i^{n-1}H_{nm}\right)\left(i^nH_{n+1,m+1}\right)^*\right]\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} iC_n\left(-b_{n+1}^{-m}H_{nm}H_{n+1,m-1}^* + b_{n+1}^{m}H_{nm}H_{n+1,m+1}^*\right)\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left(-b_{n+1}^{-m}H_{nm}H_{n+1,m-1}^* + b_{n+1}^{m}H_{nm}H_{n+1,m+1}^*\right)\right\}. \quad \text{(C1)}$$

Note that $\omega = kc$ with the sound speed in fluid c, and $\mathrm{Im}\{iX\} = \mathrm{Re}\{X\}$. By replacing $-m$ with m for the first part, Eq. (C1) is further written as

$$F_x = \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_nb_{n+1}^{m}\left(-H_{n,-m}H_{n+1,-m-1}^* + H_{nm}H_{n+1,m+1}^*\right)\right\}, \quad \text{(C2)}$$

which is Eq. (46) in Ref. 16.

The y-component of ARF is

$$F_y = \frac{\rho_0\Phi_0^2}{4}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left(b_{n+1}^{-m}a_n^m a_{n+1}^{m-1*} + b_{n+1}^{m}a_n^m a_{n+1}^{m+1*}\right)\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left[b_{n+1}^{-m}\left(i^{n-1}H_{nm}\right)\left(i^nH_{n+1,m-1}\right)^* + b_{n+1}^{m}\left(i^{n-1}H_{nm}\right)\left(i^nH_{n+1,m+1}\right)^*\right]\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} iC_n\left(-b_{n+1}^{-m}H_{nm}H_{n+1,m-1}^* - b_{n+1}^{m}H_{nm}H_{n+1,m+1}^*\right)\right\}$$
$$= \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n\left(b_{n+1}^{-m}H_{nm}H_{n+1,m-1}^* + b_{n+1}^{m}H_{nm}H_{n+1,m+1}^*\right)\right\}. \quad \text{(C3)}$$

Similar to the derivation for F_x, taking $C_n^{m\mp1} = C_n$ for a sphere and replacing $-m$ with m, F_y can also be written as

$$F_y = \frac{1}{4\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_nb_{n+1}^{m}\left(H_{n,-m}H_{n+1,-m-1}^* + H_{nm}H_{n+1,m+1}^*\right)\right\}, \quad \text{(C4)}$$

which is Eq. (47) in Ref. 16.

The z-component of ARF is

$$F_z = \frac{\rho_0\Phi_0^2}{2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m c_{n+1}^m a_n^m a_{n+1}^{m*}\right\}$$
$$= \frac{1}{2\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m c_{n+1}^m \left(i^{n-1}H_{nm}\right)\left(i^nH_{n+1,m}\right)^*\right\}$$
$$= \frac{1}{2\pi^2\rho_0k^2c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} (-i)C_n^m c_{n+1}^m H_{nm}H_{n+1,m}^*\right\}$$
$$= -\frac{1}{2\pi^2\rho_0k^2c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m c_{n+1}^m H_{nm}H_{n+1,m}^*\right\}, \quad \text{(C5)}$$

which is Eq. (48) in Ref. 16 by replacing C_n^m with C_n for a sphere.

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APPENDIX D: ANGULAR MOMENTUM AND LADDER OPERATORS

The ladder operators $L_\pm$ has the relationship with the lateral components of the angular momentum operator $L_{x,y}$: $L_\pm = L_x \pm iL_y$.[49] The recursion relations of ladder operators $L_\pm$ (or axial component of angular momentum operator L_z) and normalized spherical harmonics are[66]

$$L_+ Y_n^m = \bar{b}_n^{-m} Y_n^{m+1}, \tag{D1a}$$

$$L_- Y_n^m = \bar{b}_n^{m} Y_n^{m-1}, \tag{D1b}$$

$$L_z Y_n^m = m Y_n^m, \tag{D1c}$$

with $\bar{b}_n^m = \sqrt{(n+m)(n-m+1)}$.

APPENDIX E: DETAILED DERIVATION OF ART WITH CORRECT INDEX

1. Detailed derivation of T_x

Based on the ART formulas of Eq. (7) from Ref. 22, the expression of x-component of ART is

$$T_x = -\frac{\rho_0 \Phi_0^2}{2k} \iint_{S_0} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} Y_n^{m*} L_x Y_{n'}^{m'} \sin\theta d\theta d\varphi \right\}. \tag{E1}$$

With insertion of Eqs. (D1a) and (D1b) into Eq. (E1) and since $L_x = (L_+ + L_-)/2$,

$$\begin{aligned} T_x &= -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} \iint_{S_0} Y_n^{m*} (L_+ + L_-) Y_{n'}^{m'} \sin\theta d\theta d\varphi \right\} \\ &= -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'-1} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} \iint_{S_0} Y_n^{m*} \bar{b}_{n'}^{-m'} Y_{n'}^{m'+1} \sin\theta d\theta d\varphi \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'+1}^{n'} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} \iint_{S_0} Y_n^{m*} \bar{b}_{n'}^{m'} Y_{n'}^{m'-1} \sin\theta d\theta d\varphi \right\}. \end{aligned} \tag{E2}$$

The regime of $(n', m',)$ in the summation symbol is listed in Table III.

Using Eq. (A1), the expression of T_x is

$$\begin{aligned} T_x &= -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'-1} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} \bar{b}_{n'}^{-m'} \delta_{nn'} \delta_{m,m'+1} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'+1}^{n'} i^{n-n'} \left(a_n^{m*} + s_n^{m*}\right) s_{n'}^{m'} \bar{b}_{n'}^{m'} \delta_{nn'} \delta_{m,m'-1} \right\} \\ &= -\frac{\rho_0 \Phi_0^2}{4k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} \left(a_n^{m*} + s_n^{m*}\right) s_n^{m-1} \bar{b}_n^{-m+1} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n-1} \left(a_n^{m*} + s_n^{m*}\right) s_n^{m+1} \bar{b}_n^{m+1} \right\}. \end{aligned} \tag{E3}$$

A re-index is necessary for the second part of Eq. (E3) by using $q = m+1 \in [-n+1, n]$ and noting that $\bar{b}_n^{-m+1} = \bar{b}_n^m$, we have

TABLE III. Regime of (n', m') in normalized spherical harmonics for derivation of F_x and F_y. Note that based on the definition in Eq. (1), we have $n' \in [0, \infty]$ and $m' \in [-n', n']$.

	n'	m'	Intersection
$Y_{n'}^{m'+1}$	$n' \in [0, \infty]$	$m' \in [-n'-1, n'-1]$	$n' \in [0, \infty]$, $m' \in [-n', n'-1]$
$Y_{n'}^{m'-1}$	$n' \in [0, \infty]$	$m' \in [-n'+1, n'+1]$	$n' \in [0, \infty]$, $m' \in [-n'+1, n']$

$$T_x = -\frac{\rho_0\Phi_0^2}{4k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\left(a_n^{m*}+s_n^{m*}\right)s_n^{m-1}\bar{b}_n^{-m+1}+\sum_{n=0}^{\infty}\sum_{q=-n+1}^{n}\left(a_n^{q-1*}+s_n^{q-1*}\right)s_n^{q}\bar{b}_n^{q}\right\}$$
$$= -\frac{\rho_0\Phi_0^2}{4k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^{m}\left[\left(a_n^{m*}+s_n^{m*}\right)s_n^{m-1}+\left(a_n^{m-1*}+s_n^{m-1*}\right)s_n^{m}\right]\right\}, \quad \text{(E4)}$$

which is Eq. (14a) in Sec. VI.

2. Derivation of T_y

The expression of the y-component of ART is

$$T_y = -\frac{\rho_0\Phi_0^2}{2k}\iint_{S_0}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n-n'}\left(a_n^{m*}+s_n^{m*}\right)s_{n'}^{m'}Y_n^{m*}L_yY_{n'}^{m'}\sin\theta d\theta d\varphi\right\}. \quad \text{(E5)}$$

Similar to the derivation for T_x, the final expression of T_y in terms of a_n^m and s_n^m can be obtained by using Eqs. (D1a) and (D1b) in Eq (E5) and $L_y = (L_+ - L_-)/2i$ instead of L_x, as given in Eq. (14b) and omitted here for brevity.

3. Detailed derivation of T_z

The expression of the z-component of ART is

$$T_z = -\frac{\rho_0\Phi_0^2}{2k}\iint_{S_0}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n-n'}\left(a_n^{m*}+s_n^{m*}\right)s_{n'}^{m'}Y_n^{m*}L_zY_{n'}^{m'}\sin\theta d\theta d\varphi\right\}. \quad \text{(E6)}$$

Inserting Eq. (D1c) into Eq. (E6), we have

$$T_z = -\frac{\rho_0\Phi_0^2}{2k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n-n'}\left(a_n^{m*}+s_n^{m*}\right)s_{n'}^{m'}\iint_{S_0}Y_n^{m*}mY_{n'}^{m'}\sin\theta d\theta d\varphi\right\}. \quad \text{(E7)}$$

By substituting Eq. (A1) into Eq. (E7), the final expression of T_z in terms of a_n^m and s_n^m can be derived as

$$T_z = -\frac{\rho_0\Phi_0^2}{2k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} i^{n-n'}\left(a_n^{m*}+s_n^{m*}\right)s_{n'}^{m'}m\delta_{nn'}\delta_{mm'}\right\}$$
$$= -\frac{\rho_0\Phi_0^2}{2k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}m\left(a_n^{m*}+s_n^{m*}\right)s_n^{m}\right\}, \quad \text{(E8)}$$

which is Eq. (14c) in Sec. VI.

APPENDIX F: EQUIVALENCE OF EQ. (14) AND FORMULAS BY GONG AND BAUDOIN

By substituting Eq. (7) into Eq. (15), we can prove that the three components of ART formulas are equivalent to those in terms of H_{nm} by Gong and Baudoin [see Eqs. (10)–(12) in Ref. 17], respectively. The detailed derivations are given below. The x-component of ART is

$$T_x = -\frac{\rho_0\Phi_0^2}{4k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^{m}\bar{C}_n^{m}a_n^{m*}a_n^{m-1}\right\}$$
$$= -\frac{1}{4\pi^2\rho_0k^3c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^{m}\bar{C}_n^{m}\left(i^{n-1}H_{nm}\right)^*\left(i^{n-1}H_{n,m-1}\right)\right\}$$
$$= -\frac{1}{4\pi^2\rho_0k^3c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^{m}\bar{C}_n^{m}H_{nm}^*H_{n,m-1}\right\}, \quad \text{(F1)}$$

which is Eq. (10) in Ref. 17.

The y-component of ART is

$$T_y = -\frac{\rho_0\Phi_0^2}{4k}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^m\bar{C}_n^m a_n^{m*}a_n^{m-1}\right\}$$
$$= -\frac{1}{4\pi^2\rho_0 k^3 c^2}\mathrm{Im}$$
$$\times\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^m\bar{C}_n^m\left(i^{n-1}H_{nm}\right)^*\left(i^{n-1}H_{n,m-1}\right)\right\}$$
$$= -\frac{1}{4\pi^2\rho_0 k^3 c^2}\mathrm{Im}\left\{\sum_{n=0}^{\infty}\sum_{m=-n+1}^{n}\bar{b}_n^m\bar{C}_n^m H_{nm}^* H_{n,m-1}\right\}, \tag{F2}$$

which is Eq. (11) in Ref. 17.

The z-component of ART is

$$T_z = -\frac{\rho_0\Phi_0^2}{2k}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}m\bar{D}_n^m a_n^{m*}a_n^{m}\right\}$$
$$= -\frac{1}{2\pi^2\rho_0 k^3 c^2}\mathrm{Re}$$
$$\times\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}m\bar{D}_n^m\left(i^{n-1}H_{nm}\right)^*\left(i^{n-1}H_{nm}\right)\right\}$$
$$= -\frac{1}{2\pi^2\rho_0 k^3 c^2}\mathrm{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=-n}^{n}m\bar{D}_n^m H_{nm}^* H_{nm}\right\}, \tag{F3}$$

which is Eq. (12) in Ref. 17.

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