Acoustic resonance scattering of Bessel beam by elastic spheroids in water

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Abstract

The acoustic resonance scattering of a zero-order Bessel beam by elastic spheroids immersed in water and centered along the beam axis is investigated. The T-matrix method is utilized to establish the Bessel acoustic scattering formulas through the harmonic expansion of Bessel beam. For a specific half-conical angle β , the far-field backscattering form functions of spheroids with different materials and a variety of aspect ratios are curved versus the dimensionless frequency $kL/2$, where k is the wave number in water and L is the length of the spheroid. By subtracting an appropriate background from the total backscattering form functions, the corresponding resonance of the elastic spheroids can be determined. It is concluded that the T-matrix method is effective to calculate both the total and resonance backscattering fields under an end-on incident Bessel beam illumination. Moreover, by selecting appropriate half-conical angles, the excitation of certain resonance of elastic spheroids may be suppressed and this phenomenon may have some potential value in practical applications.

Keywords: Acoustic resonance scattering, Zero-order Bessel beam, T-matrix method, Backscattering form function, Elastic spheroid

1. Introduction

During the past decades, considerable efforts to analyze acoustic resonance scattering by elastic targets immersed in an ideal fluid illuminated by plane wave has been devoted through experimental and theoretical methods by many authors[Flax et al. (1978); Gaunaurd and Uberall (1983); Werby et al. (1988); Williams and Marston (1986); Bao et al. (1992); Haumesser et al. (2002)]. When the incident acoustic excitation takes Bessel beam into consideration, the corresponding investigation is very limited. Bessel beam was first introduced by Durnin et al [Durnin (1987); Durnin et al. (1987)] in optics and then had a following development in acoustics [Campbell and Soloway (1990); Lu and Greenleaf (1990); Marston (1992)]. Bessel beam is characterized by an important parameter, termed as half-conical angle β, which describes the angle of the planar wave components of the beam relative to the beam axis. Note that in practice, Bessel beam can only propagate over a limited distance without spreading due to the finite width of sources, however, in the published studies, the ideal Bessel beam is always taken as the incident acoustic excitation. Also, in our work, targets are under the illumination of ideal Bessel beam.

Recently, Bessel beam has attracted increasing attention in acoustic aspect because it has demonstrated several advantages over the plane waves, such as the characteristics of non-diffraction and the ability to retain its form without block. Marston first studied the scattering characteristics of both rigid and soft spheres centered on a Bessel beam and curved the form function modulus as a function of scattering angle θ [Marston (2007a)]. After that in 2007, Marston continued to investigate the resonance scattering of elastic solid sphere and spherical shell under the Bessel beam illumination and evaluated the influences of several selective half-conical angles on the suppression of backscattering resonance [Marston (2007b)]. Subsequently, further study on acoustic scattering associated with Bessel beam was also reported by others, such as Mitri and Li's research group. There has been considerable interest in Mitri's study of high-order Bessel beams to explore acoustic scattering characteristics by several objects, including rigid (movable and immovable) spheres [Mitri (2009a; 2011); Mitri and Silva (2011)], elastic spheres [Mitri (2008a; 2009b)], and elastic spherical shells [Mitri (2010; 2012)]. In addition, Mitri devoted much efforts to calculate acoustic radiation force of spheres and rigid spheroids [Mitri (2008b; 2009c; 2015); Silva et al. (2013)], which may provide an impetus to design acoustic tweezers. Moreover, our research group has studied the scattering properties of arbitrary-shape rigid scatterer facing the incident Bessel beam. In our work, the backscattering fields of rigid spheroid and finite cylinder with two hemispherical endcaps were investigated, and the peak to peak intervals in backscattering form functions were analyzed both in geometry and using numerical method [Li et al. (2015)].

Despite the recent reports about acoustic Bessel beam reviewed above, to date, it still remains an unsolved problem when calculation of resonance scattering by elastic spheroid placed in Bessel beam is taken into consideration. In previous studies published by Marston and Mitri, the exact scattering by spheres and spherical shells was expressed as a partial wave series. Specifically, Mitri improved the partial-wave series expansion (PWSE) method to calculate acoustic radiation force of rigid spheroid under the illumination of Bessel beam [Mitri (2015)]. Unfortunately, the PWSE method has not yet been further developed to provide theoretical analysis on elastic spheroid interacted with Bessel beams. Instead, the T-matrix method, as originally conceived by Waterman [Waterman (1965; 1971)], has been demonstrated a very efficient tool to handle acoustic scattering problems by elastic targets with arbitrary shape, for instance, cylinders [Varadan 1978], spheres [Pao and Mow (1963)] and spheroids [Flax et al. (1983); Bostrom (1980a)]. The philosophy of the T-matrix method is to expand all field quantities in terms of a set of spherical functions in order to obtain the T matrix (also called transition matrix) that relates the known coefficients of expansion of the incident wave to the unknown expansion coefficients of the scattered field. When all parameters of scatterer and incident wave are provided, the scattered fields can be obtained immediately by using the T-matrix method. To our knowledge, there is no evidence that the Bessel beam scattering by elastic spheroid immersed in an ideal fluid has been studied in previous work published. To this end, in the present study, we aim to extend the application of T-matrix to study the acoustic scattering characteristics of elastic spheroid under the illumination of Bessel beam.

This paper is outlined as follows. In section 2, a brief review of the T-matrix method for acoustic scattering is given and subsequently, we derive the formula of incident coefficients of Bessel beam. In section 3, two numerical examples are carried out to explore the acoustic resonance characteristics of PMMA sphere and spheroid immersed in an ideal fluid under the illumination of ideal zero-order Bessel beam. Finally, the conclusion of this paper is conducted in section 4.

2. Theoretical formulation

In this section, the theoretical formulas of an elastic scatterer immersed in an ideal fluid under the normal illumination of Bessel beam by using the well-known T-matrix method are present. As in the previous work of other authors, the ideal Bessel beam is also considered here.

2.1 A brief review of the T-matrix method for the scattering

To determine the transition matrix, which connects the expansion coefficients of the incident and scattered fields, the integral representations for the displacement field, the boundary conditions and expansions of surface fields should be provided explicitly. Now consider an elastic scatterer with its geometry shown in Fig. 1. The boundary surface between fluid and solid is denoted by *S*. The host medium is homogeneous water with density ρ_f and lame parameter λ_f. The properties of elastic scatterer are given by density ρ and lame parameters λ and μ. For convenience, we would rather use the vector formalism than the scalar wave equation for both the fluid (outside *S*) and solid (inside *S*) regions to facilitate the application of the boundary conditions. As given in detail by Bostrom and Peterson [Bostrom (1980b); Peterson et al. (1980]], here, we will only present the most pertinent formulas.

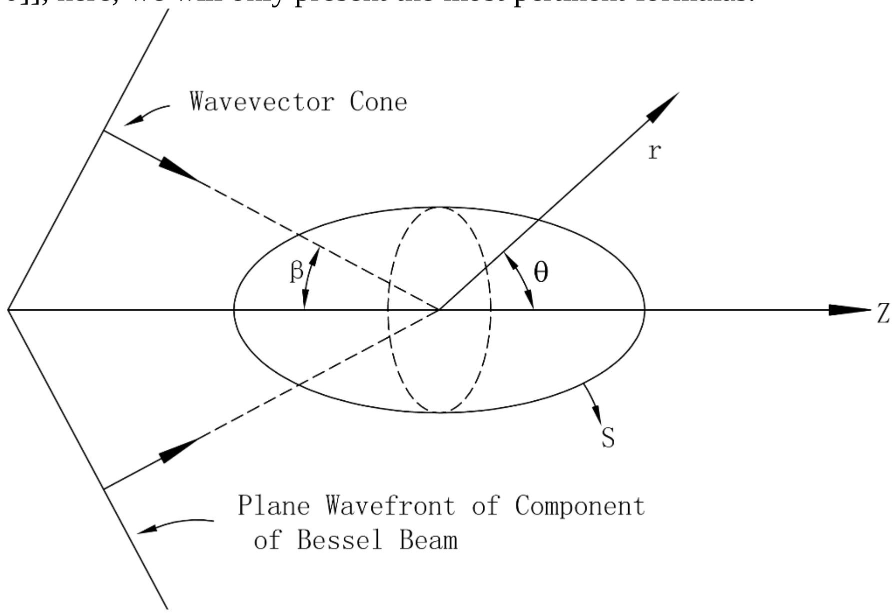

Fig. 1 Elastic scatterer under Bessel beam illumination.

The starting point for the following is the integral representation for the displacement $\mathbf{u} = \mathbf{u}^i + \mathbf{u}^s$ in the fluid region

$$\int_S \left\{ \mathbf{u}_+ \bullet \left[n \bullet \mathbf{\Sigma}_f(\mathbf{r},\mathbf{r}') \right] - \mathbf{G}_f(\mathbf{r},\mathbf{r}') \bullet \mathbf{t}_+ \right\} \mathrm{d}S = \begin{cases} \mathbf{u}^S & \mathbf{r} \text{ outside } S \\ -\mathbf{u}^i & \mathbf{r} \text{ intside } S \end{cases} \tag{1}$$

Here n is the unit normal taken as outward pointing, $\mathbf{\Sigma}_f$ and $\mathbf{G}_f$ are the free space Green's stress triadic and Green's dyadic, respectively. The surface traction vector $\mathbf{t}_+$ can be obtain by Hooke's law expressed as

$$\mathbf{t} = n\bullet\left[\lambda\mathbf{I}\nabla\bullet\mathbf{u} + \mu\left(\nabla\mathbf{u} + \mathbf{u}\nabla\right)\right] \tag{2}$$

For the solid region inside *S*, the displacement **u** is governed by the following representation

$$\int_S \left\{\mathbf{u}_- \bullet \left[n\bullet\boldsymbol{\Sigma}(\mathbf{r},\mathbf{r}')\right] - \mathbf{G}(\mathbf{r},\mathbf{r}')\bullet\mathbf{t}_-\left(\mathbf{r}'\right)\right\}\mathrm{d}S = \begin{cases} -\mathbf{u} & \mathbf{r}\text{ inside } S \\ 0 & \mathbf{r}\text{ outside } S \end{cases} \tag{3}$$

where $\mathbf{G}$ is the Green's displacement dyadic and $\boldsymbol{\Sigma}$ is the Green's stress dyadic related to $\mathbf{G}$ by Hooke's law [Varadan (1980)]. For an elastic scatterer immersed in water, the relevant boundary conditions on surface are given by

$$\begin{aligned} n\bullet\mathbf{u}_+ &= n\bullet\mathbf{u}_- \\ n\bullet\mathbf{t}_+ &= n\bullet\mathbf{t}_- \\ n\times\mathbf{t}_- &= 0 \end{aligned} \tag{4}$$

Both the integral representations for the displacement fields and the corresponding boundary conditions are presented above. Subsequently, the incident and scattered displacement fields are expanded in vector spherical functions, with the time factor e^{-iwt} being suppressed throughout, as follows:

$$\mathbf{u}^i = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\sigma} a_{nm\sigma}\,\mathrm{Re}\phi_{nm\sigma}\left(\mathbf{r}\right) \tag{5}$$

$$\mathbf{u}^s = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\sigma} f_{nm\sigma}\phi_{nm\sigma}\left(\mathbf{r}\right) \tag{6}$$

Finally, the incident and scattered field coefficients are related through the transition as given by

$$f_{nm\sigma} = \sum_{n'm'\sigma'} T_{nm\sigma,n'm'\sigma'}a_{n'm'\sigma'} \tag{7}$$

where

$$T = -\mathrm{Re}\left[QR^{-1}P\right]\left[QR^{-1}P\right]^{-1} \tag{8}$$

The detailed expressions of matrices Q, R and P are given by Bostrom and Peterson [Bostrom (1980b); Peterson et al. (1980)]. Through the formulas described above, the unknown scattered coefficient can be computed immediately, and thus the scattered field can be acquired correspondingly.

2.2 Derivation of the incident coefficients of Bessel beam

In this part, the formulas of the incident coefficient $a_{nm\sigma}$ in the case of Bessel beam will be derived for the combination with original T-matrix method. Note that by using the T-matrix method, all field quantities should be expanded in terms of one series of spherical functions. Here the following scalar basis functions adopted in the computation are defined as

$$\phi_{nm\sigma}\left(\mathbf{r}\right) = \xi_{nm}h_n\left(kr\right)P_n^m\left(\cos\theta\right)\begin{pmatrix}\cos\left(m\varphi\right), & \sigma = \mathrm{e} \\ \sin\left(m\varphi\right), & \sigma = \mathrm{o}\end{pmatrix} \tag{9}$$

$$\xi_{nm} = \left(\varepsilon_m\frac{\left(2n+1\right)\left(n-m\right)!}{4\pi\left(n+m\right)!}\right)^{1/2} \tag{10}$$

where $h_n(kr)$ is the spherical Hankle function of the first kind, P_n^m is the associated Legendre function and $\varepsilon_m = 2-\delta_{m0}$ is the Neumann factor. $\sigma = \mathrm{e, o}$ $(\mathrm{even, odd})$ specifies azimuthal parity, $m = 0,1,\cdots,n$ specifies rank and $n = 0,1,\cdots$specifies order of the spherical harmonics. Also, θ and φ are the axial and azimuthal angles, respectively.

The wave function defined above may be used to expand the scattered fields that satisfy radiation conditions at infinity. But to expand the incident fields that are finite at origin, the spherical Hankle function in the wave function in Eq. (9) should be replaced by the spherical Bessel function, which is regular at origin. In the case of the ideal zero-order Bessel beam we are interested, the expression of incident coefficients can be derived by using the foregoing basis functions in the following procedure.

Provided that the complex velocity potential of ideal zero-order Bessel beam could be denoted as follows [Marston (2007a)]

$$\phi_B(z,\rho) = \phi_0 \exp(i\kappa z) J_0(\mu\rho) \tag{11}$$

where ϕ_0 stands for the beam amplitude, z and ρ specify the axial and radial coordinates, κ and μ, satisfying the relation $\kappa^2+\mu^2=k^2$, represent the axial and radial wavenumbers, and J_0 is a zero-order Bessel beam.

Given by Eq. (B2) in APPENDIX B by Marston [Marston (2007a)], Eq. (11) may be expanded in spherical partial wave as

$$\phi_B(r,\theta) = \phi_0 \sum_{n=0}^{\infty} i^n \times (2n+1) j_n(kr) P_n(\cos\theta) P_n(\cos\beta) \tag{12}$$

By using the addition theorem of the Legendre polynomial [Stratton (2007)], the desired integration can be obtained

$$P_n(\cos\theta) = \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} \times P_n^m(\cos\theta_i) P_n^m(\cos\theta_s) \cos m(\varphi_i - \varphi_s) \tag{13}$$

where θ_i and θ_s represent the incident and scattered axial angles, respectively.

Combining Eq. (9), (10), (12) and (13) gives an expression of the incident coefficients $a_{nm\sigma}$ under ideal zero-order Bessel beam as follows

$$a_{nm\sigma} = 4\pi \xi_{nm} i^n \times P_n^m(\cos\theta_i) P_n^m(\cos\beta) \begin{pmatrix} \cos(m\varphi), & \sigma = \mathrm{e} \\ \sin(m\varphi), & \sigma = \mathrm{o} \end{pmatrix} \tag{14}$$

It should be noted that the expression of the incident coefficients depend on incident axial angles, half-conical angle and incident azimuthal angle. The ideal Bessel beam taken as the incident excitation is with unit amplitude $(\phi_0 = 1)$ and meanwhile normally incident $(\theta_i = 0)$. The half-conical angle β is decided by plane wavefront of component of Bessel beam relative to the beam axis. Specially, $\beta = 0$ gives the limiting case of an ordinary plane wave. During the derivation process presented above, the azimuthal angle φ is neglected, which is totally feasible for rotationally symmetrical targets considered here.

Substituting Eq. (8) and (14) into the relation given by Eq. (7), the scattered fields of elastic targets under the illumination of ideal zero-order Bessel beam can be obtained and this may help lay a foundation for further exploring the characteristics of acoustic scattering of Bessel beam.

3. Numerical results

In order to study the Bessel beam modification to the coupling to resonances of elastic scatterers, several numerical examples are presented by using T-matrix method. In the first part, sphere model is given to verify the validity of the T-matrix method to investigate the acoustic resonance scattering under the illumination of Bessel beam. While in the second part, we extend the range of applicability of T-matrix to further investigate Bessel beam scattering characteristics of spheroid.

3.1 Acoustic Resonance Scattering from Spheres

In this part, acoustic resonance analyses for the case of Bessel beam scattering from polymethylmethacrylate (PMMA) sphere are carried out and the ambient ideal fluid here is considered to be water. The corresponding material parameters in our numerical example are listed explicitly in Table 1. To facilitate the discussion in the following, several half-conical angles are selected specially which are defined β_n as the lowest root of $P_n\left(\cos\beta_n\right)=0$. The 6-digit approximations to β_n for $n=2,3,4$ and 5 are $\beta_2=54.7346^\circ$, $\beta_3=39.2315^\circ$, $\beta_4=30.5556^\circ$ and $\beta_5=25.0173^\circ$, respectively.

The backscattering form functions with different half-conical angles are calculated and plotted in Fig. 2. The black solid curve shown in Fig. 2 is the backscattering ($\theta=\pi$ in Eq. (9)) for plane wave illumination $\left(\beta=0\right)$. When other approximations $\left(\beta_2,\beta_3,\beta_4 \text{ and } \beta_5\right)$ are implemented, it could be found that the nth order resonance is suppressed. This is most easily seen from the dashed black line which has $\beta=\beta_2$. This behavior is in agreement with Eq. (14) because of the dependence on $P_n\left(\cos\beta\right)$. All of the curves shown in Fig. 2 agree very well with results in Marston's work.

Table 1 Material parameters

Material	Density (kg/m^3)	Longitudinal velocity (m/s)	Shear velocity (m/s)
PMMA	1.19	2690	1340
Water	1000	1500	—

3.2 Acoustic Resonance Scattering from Spheroids

In the prior part, backscattering form function of PMMA sphere with different half-conical angles are calculated. In the following part, we will further investigate the influences of different half-conical angles upon the backscattering resonance of PMMA spheroid. The aspect ratio (semi-minor axis/semi-major axis) of spheroid is 0.95. The backscattering form functions of PMMA spheroid under the illumination of Bessel beam with different half-conical angles are plotted in Fig. 3. The plane wave case is also given here by the black solid curve when $\beta=0^\circ$. Similarly, by choosing approximations $\left(\beta_2,\beta_3,\beta_4 \text{ and } \beta_5\right)$, the corresponding resonances are suppressed. That is, in the case of $\beta=\beta_2$, the 2nd order resonance is suppressed, in the case of $\beta=\beta_3$, the 3rd order resonance is suppressed, and for the remaining β_4,β_5, the same results can be obtained.

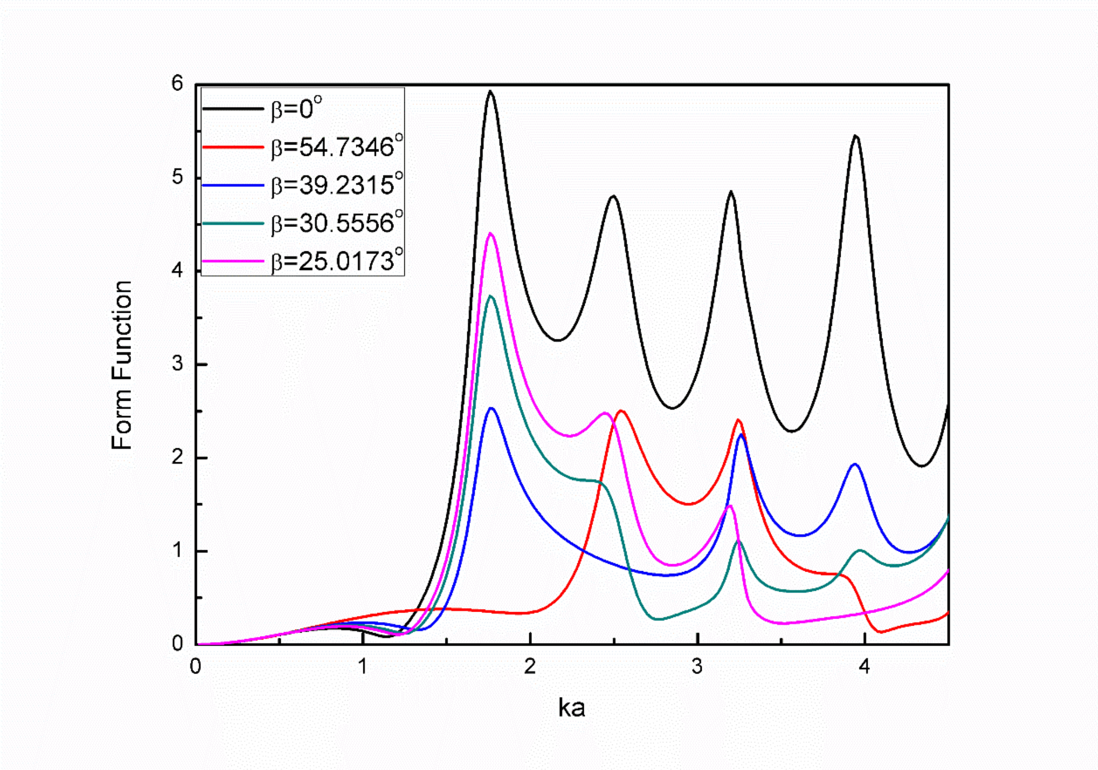

Fig. 2 Backscattering form function modulus computed as a function of dimensionless frequency *ka* for an elastic PMMA sphere in water under the illumination of Bessel beam with different half-conical angles

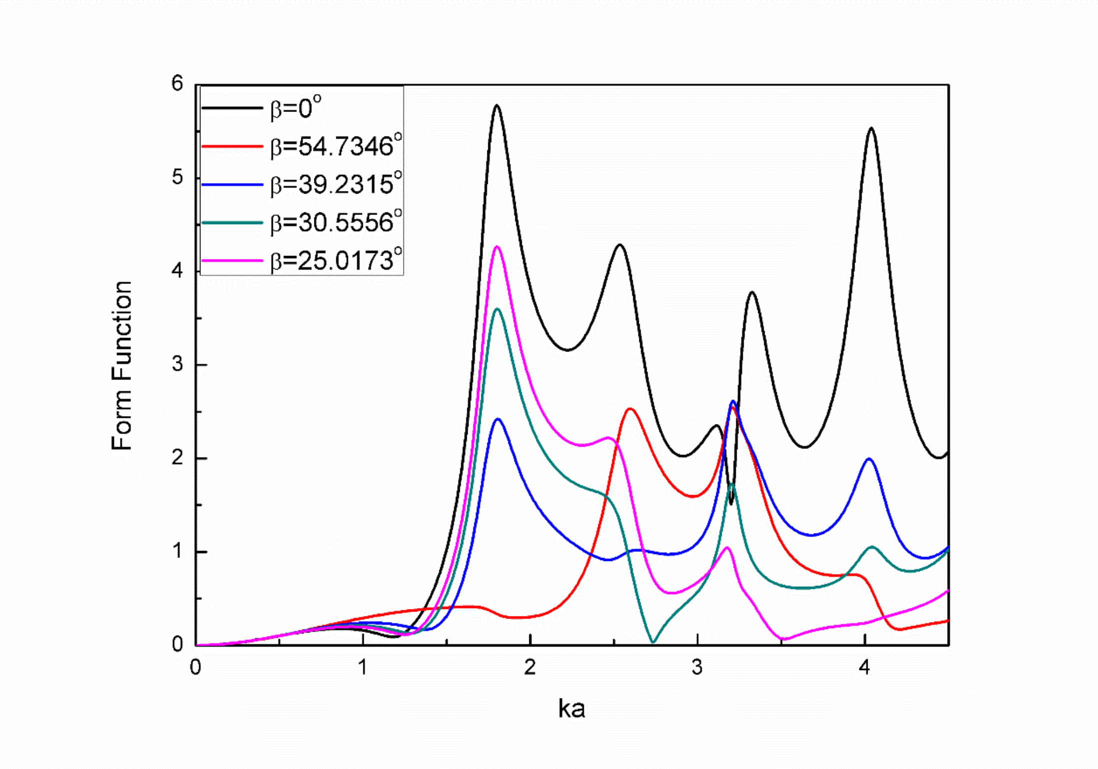

Fig. 3 Backscattering form function modulus computed as a function of dimensionless frequency *ka* for an elastic PMMA spheroid in water under the illumination of Bessel beam with different half-conical angles

4. Conclusions

In this paper, two numerical examples are carried out to calculate the backscattering form functions of underwater PMMA sphere and spheroid illuminated by zero-order Bessel beam with several selective half-conical angles. The T-matrix method is implemented and thus demonstrated to be an effective tool to compute the scattered fields. Also, the T-matrix method is able to expand its range of applicability through the combination with Bessel beam. With appropriate selection of specific Bessel beam parameters, some resonances can be suppressed and this in turn may provide useful directions on engineering applications.

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